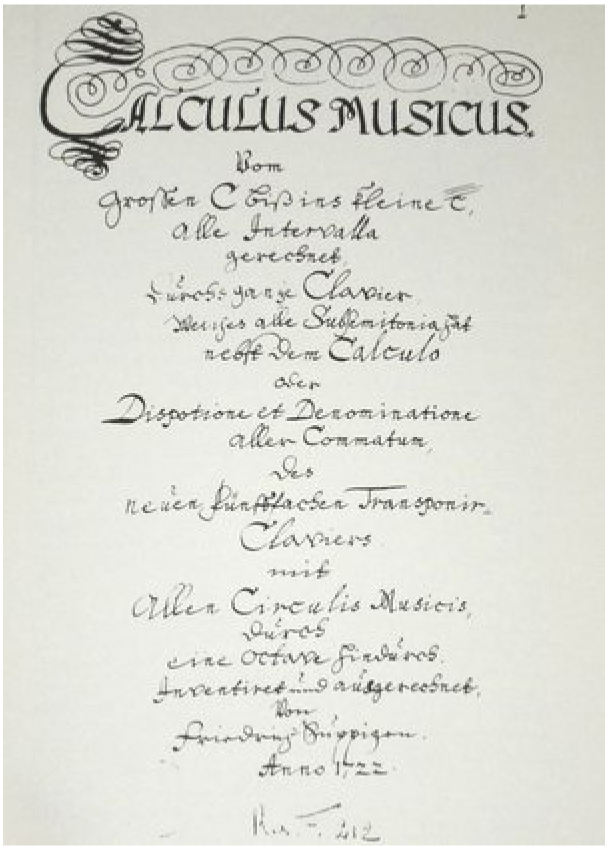
Friedrich Suppig's

Diagrammatic Tunings

CH 3072 Ostermundigen



This page intentionally blank

Abstract

An earlier paper, “The Esoteric Keyboard Temperaments of J. S. Bach” [1], described a new tuning interpretation of the spiral on the cover sheet of Johann Sebastian Bach’s *Das Wohltemperirte Clavier*. In the current article, the approach is reapplied to the spirals occurring on the cover sheets of Friedrich Suppig’s *Labyrinthus musicus* and *Calculus musicus*. Results are presented and comparison is made with previous findings.

Introduction

Friedrich Suppig's *Labyrinthus musicus* and *Calculus musicus*, dated Dresden 24 June 1722, can be found in the Paris Bibliotheque du Conservatoire (Res. F211-212, acquired in 1894). A facsimile of both documents was published in 1990 by the Diapason Press [2] with a comprehensive introduction by the editor Rudolf Rasch.

The *Labyrinthus musicus* consists of a 'Fantasia' for harpsichord or organ composed using all 24 keys, while the *Calculus musicus* is a short theoretical treatise on tuning. The two are believed to form a unity. Johann Mattheson mentioned the *Labyrinthus musicus* in his *Crtitica musica* (1722), describing Suppig as an organist in the Dresden suburbs. In 1863, the document was in possession of Louis Kindscher of Cöthen who mentioned the work in the periodical *Euterpe* (22, 1863).

The manuscript contains a dedication to the government of Dresden. Suppig alludes in his dedication to:

> ...many able men that have enriched musical harmony with their wondrous inventions and have bought this art almost to its summit from which it is sufficient to name only a few as honourable examples: namely Kuhnau, Krieger, Vetter, Buttstett, Pezold and several others.

He goes on to say:

> I do not have to be ashamed when - by order of and persuaded by a man who has great authority to me - I offer and dedicate these little works of mine as a sign of gratitude towards your favours...

Suppig does not name the authority, however.

Although, the manuscripts were dedicated to the government of Dresden, the surving documents surfaced in Cöthen in the 19th century, the town where Johann Sebastian Bach was located in 1722. It follows that either they did not become the property of the city of Dresden or that they were returned to Suppig for some reason or else they are copies.

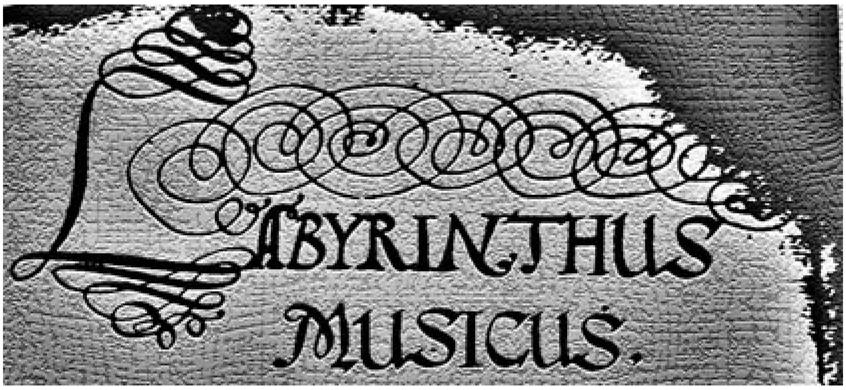
Commenting on the *Calculus musicus*, Rasch observes [2] "Rather surprisingly, it contains discussions of two tuning systems that one would not expect in such circumstances: a 19-tone just intonation and the 31 tone system (multiple division.)"

In Suppig's 19-tone just intonation, there are thirteen pure fifths, five comatic fifths and one that is flat by nearly 30 cents. Some of the major thirds are wide by some 41 cents. Suppig wrote in the preface to his *Calculus musicus* that his system was intended for a broken keyboard (*ein gebrochen Clavier*) with 20 keys per octave. He suggested that if such a keyboard is not available, resort can be made to ordinary 13-tone keyboards by replacing enharmonic tones with available enharmonic equivalents. Rasch questions whether Suppig's *Labyrinthus musicus* could be reasonably performed in this manner, however [2]. He also notes that just intonation systems with twelve or more tones per octave were never intended for practical use, but instead served as theoretical frameworks.

In his *Calculus musicus*, Suppig provides a second theoretical system, the 31-tone tuning, where the octave is divided into 31 equal tones. His description is entirely verbal, however. After describing two types of semitone, he suggests tuning 12-tone keyboards with seven major and five minor semitones. Rasch points out that if this is done, the result can only be Meantone tuning and that in such a system eight major and eight minor triads are in tune, while the remaining four major and minor triads include a wolf fifth as well as major and minor wolf thirds that are unusable in normal practice [2]. Based on his analysis of Suppig's exoteric text, Rach concludes:

> Suppig's writings strongly create the impression that he did not grasp the mathematical intricacies of equal temperament, either 12- or 31-tone just intonation. The only numerical realisations of tuning systems are those of his 19-tone just intonation, which involves only simple arithmetic. The description of the 31-tone system is entirely verbal, like that of Bulyowsky. Every reference to the mathematics required to calculate the string lengths of a 31-tone monochord is absent in the work of both authors.

As will be seen shortly, two further tuning systems are explicitly represented in diagrammatic form and placed prominently at the top of each cover sheet. Both yield fully circular temperaments.



The spiral of the *Labyrinthus musicus* suggests a circle, being elevated in the middle, curving back on itself at the far left and turning downwards at the right. Information has been lost at the right end of the circle, but by attempting to draw what remains it soon becomes apparent what is missing.

The end of the circle is depicted bottom left by a small loop (compare to the ends of the Bach spiral). A matching loop forms the serif of the letter A, making it look like Ab. Proceeding from the Ab to the left of the spiral there is a break, the next point on the circle cannot be Eb, but must be C#. Accordingly, the circle proceeds clockwise towards the flats: Ab, C#, F#, B, E, A, D, G, C, F, Bb, Eb.

Suppig's tuning scheme is minimalist, showing only needed tempering operations including necessary pure fifths, i.e., pure fifths that must be shown for the correct location of the tempering operations. Accordingly, the two pure fifths at the end are not indicated and must be added.

To exist, a loop must have one or more windings. In addition to the existential outer winding, it may have further inner windings. The term winding index is introduced as a count of the total number of loops (outer + inner). The *Labyrinthus musicus* contains four types of loops with winding indices of 1, 2, 3 and 4, respectively.

To derive the beats per second represented by a loop, remove the existential outer winding from the loop. In other words, subtract one from the winding index. Accordingly, a winding index of 1 is a pure interval, a winding index of 2 denotes an interval that beats once per second, a winding index of 3 denotes an interval that beats twice per second, while a winding index of 4 denotes an interval that beats three times per second. It can be shown by mathematics that all beat relations occur in the keyboard range C to B in the one-line octave.

The winding indices, beat rates and the reading are as follows:

Winding Indices: 1 2 3 4 3 3 3 3 2

Beat Rates: 0 1 2 3 2 2 2 2 1

Reading: Ab-x-C#-0-F#-1-B-2-E-3-A-2-D-2-G-2-C-2-F-1-Bb[-0-Eb-0-]

Harmonic Relations

Let f0,…,f11 denote twelve contiguous semitones on the keyboard. Then the fifths are f0f7, f1f8, f2f9, f3f10, f4f11, while the fourths are f0f5, f1f6, f2f7, f3f8, f4f9, f5f10, f6f11. Instead of tempering fourths, the upward inversion can be tuned as a fifth, since this beats at the same rate. With the needed octave leaps, the circle of fifths becomes: f0f7, f7f14, f2f14, f2f9, f9f16, f4f16, f4f11, f11f18, f6f18, f6f13, f1f13, f1f8, f8f15, f3f15, f3f10, f10f17, f5f17, f5f12, f0f12. Based on the results with the Bach spiral, correspondences between f0,…,f11 and the notes of the keyboard are as hypothesised in Table 1.

f0	C
f1	C#
f2	D
f3	Eb
f4	E
f5	F
f6	F#
f7	G
f8	Ab
f9	A
f10	Bb
f11	B

Table 1: correspondence between frequencies and notes

The beat rate of any fifth is given by the difference in frequency between the third harmonic of the lower note and second harmonic of the higher one. Applying the *Labyrinthus musicus* beat frequencies to the circle of fifths yields the harmonic relations in Table 2. These relationships can be combined using elementary algebra to derive the pitches f0,…,f11.

3f0 - 2f7 = 2
3f7 - 2f14 = 2
2f2 - f14 = 0
3f2 - 2f9 = 2
3f9 - 2f16 = 3
2f4 - f16 = 0
3f4 - 2f11 = 2
3f11 - 2f18 = 1
2f6 - f18 = 0
3f6 - 2f13 = 0
2f1 - f13 = 0
3f1 - 2f8 = x
3f8 - 2f15 = 0
2f3 - f15 = 0
3f3 - 2f10 = 0
3f10 - 2f17 = 1
2f5 - f17 = 0
3f5 - 2f12 = 2
2f0 - f12 = 0

Table 2: harmonic relations between fifths for Labyrinthus musicus

Derivations

The results show that tuning takes place in the one-line octave from C to B (see Table 3), an identical tuning range to the Bach spiral [1]. In the case where the beat rate of AbC# is pure, this is Cornet-Ton pitch.

Pitch of A	415 Hz	458.113 Hz	465 Hz
Beat Rate AbC#	-1.10163 Hz (wide)	0 Hz (pure)	0.175983 Hz (narrow)

Table 3: derived relation of beat rate of Ab to C# to pitch of A

As can be seen, the beat rate of AbC# changes according to pitch. For this reason an exact value cannot be indicated in Suppig's *Laybrinthus musicus* spiral, given his stated intent to allow performance on both harpsichord and organ (note, the spiral of *Das Wohltemperirte Clavier* solves such pitch issues in a more elegant fashion by providing matching transpositions at Cammerton and Cornet-Ton pitch [1]). The analysis continues for the case where AbC# is pure with the derived cents shown in Table 4 and width of thirds shown in Table 5.

C	0
C#	94.5765
D	196.854
Eb	298.486
E	393.206
F	501.211
F#	592.621
G	697.725
Ab	796.531
A	895.034
Bb	1000.44
B	1091.79

Table 4: derived cents for Laybrinthus musicus

	Min	*Maj*
C	298	393
G	303	394
D	304	396
A	305	400
E	305	403
B	305	407
F#	302	408
C#	299	407
Ab	295	403
Eb	294	399
Bb	294	396
F	295	394

Table 5: derived width of thirds in cents for Laybrinthus musicus

Winding Indices: 2 1 1 3 3 3 3 3 3 3

Beat Rates: 1 0 0 2 2 2 2 2 2 2

Reading: Ab-1-Eb-0-Bb-0-F-2-C-2-G-2-D-2-A-2-E-2-B-2 -F#[-0-C#-0-]

Applying the *Calculus musicus* beat frequencies to the circle of fifths yields the harmonic relations in Table 6.

3f0 - 2f7 = 2
3f7 - 2f14 = 2
2f2 - f14 = 0
3f2 - 2f9 = 2
3f9 - 2f16 = 2
2f4 - f16 = 0
3f4 - 2f11 = 2
3f11 - 2f18 = 2
2f6 - f18 = 0
3f6 - 2f13 = 0
2f1 - f13 = 0
3f1 - 2f8 = 0
3f8 - 2f15 = 1
2f3 - f15 = 0
3f3 - 2f10 = 0
3f10 - 2f17 = 0
2f5 - f17 = 0
3f5 - 2f12 = 2
2f0 - f12 = 0

Table 6: harmonic relations between fifths for Calculus musicus

Derivations

The results show that tuning takes place in the one-line octave from C to B, the tuning range used by *Das Wohltemperirte Clavier* [1] and the *Labyrinthus musicus*. The derived pitch of A is 458.26 Hz, within 0.2 Hz of the tuning pitch of the *Labyrinthus musicus* spiral. The derived cents are shown in Table 7, while the width of the thirds is shown in Table 8.

C	0.00
C#	94.72
D	196.86
Eb	297.30
E	394.47
F	501.21
F#	592.77
G	697.73
Ab	796.68
A	895.04
Bb	999.26
B	1093.06

Table 7: derived cents for Calculus musicus

	Min	*Maj*
C	297	394
G	302	395
D	304	396
A	305	400
E	303	402
B	304	404
F#	302	406
C#	300	406
Ab	296	403
Eb	295	400
Bb	295	398
F	295	394

Table 8: derived width of thirds in cents for Calculus musicus

Note that the increase/decrease in the width of major thirds is monotonic, while minor thirds are near-monotonic. Such features, found also in the *Laybrinthus musicus* and *Das Wohltemperirte Clavier* [1], appear indicative of purposeful design.

Comparisons

The *Labyrinthus musicus* temperament has major thirds in the range 393 to 408 cents, minor thirds from 294 to 305 cents and fifths ranging from 698 to 702 cents. The temperament of the *Calculus musicus* avoids the Pythagorean major third, confining to 394 to 406 cents. It also improves the worst minor third, restricting to the range 295 to 305 cents. The fifths range from 698 to 702 cents. Both temperaments have a Werckmeister-like shape [3] with a best major third on C and a worst major third on F#. The primary difference is the inequality of the thirds, the *Labyrinthus musicus* being the spicier option, favouring common keys. Comparisons are given in Figure 1 and Figure 2.

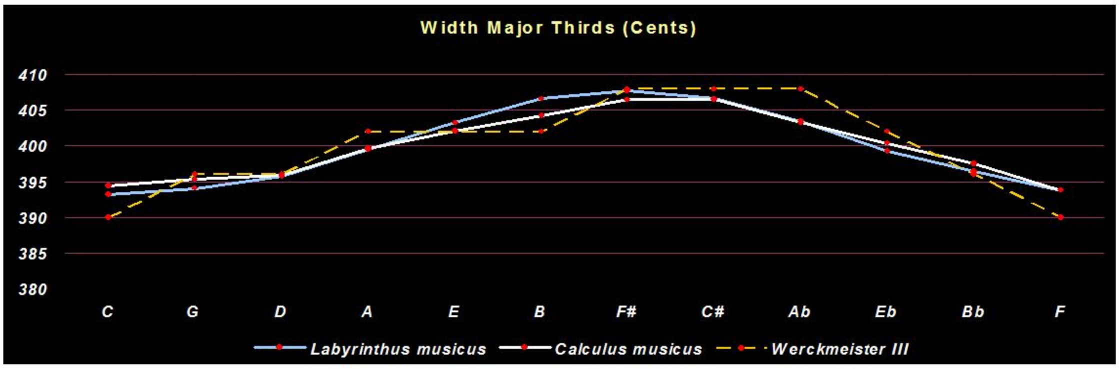


Figure 1: comparison of width of major thirds

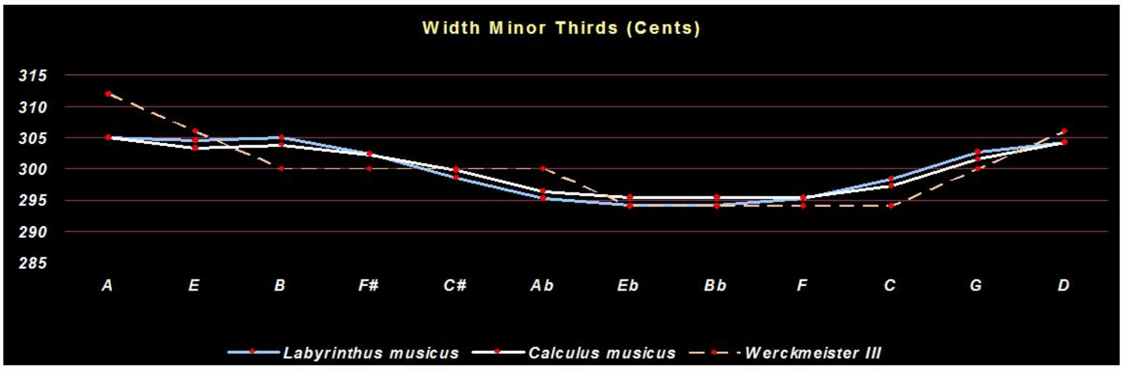
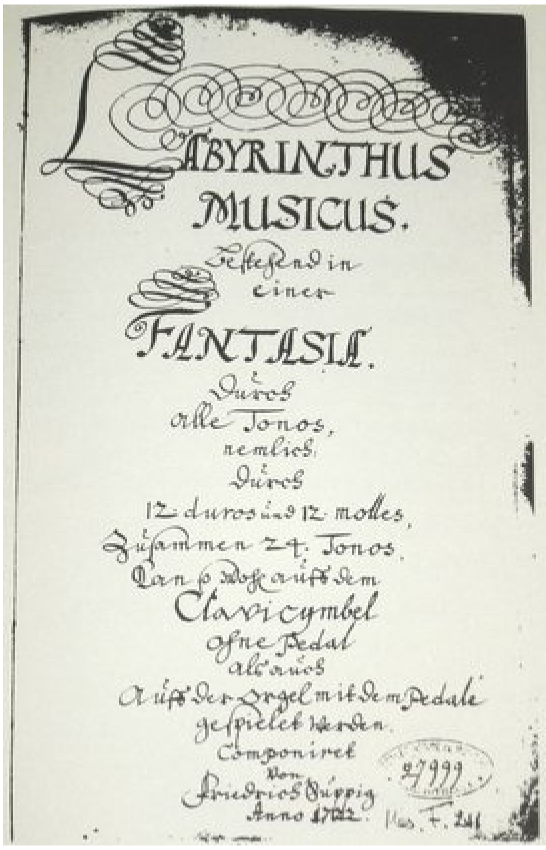


Figure 2: comparison of width of minor thirds

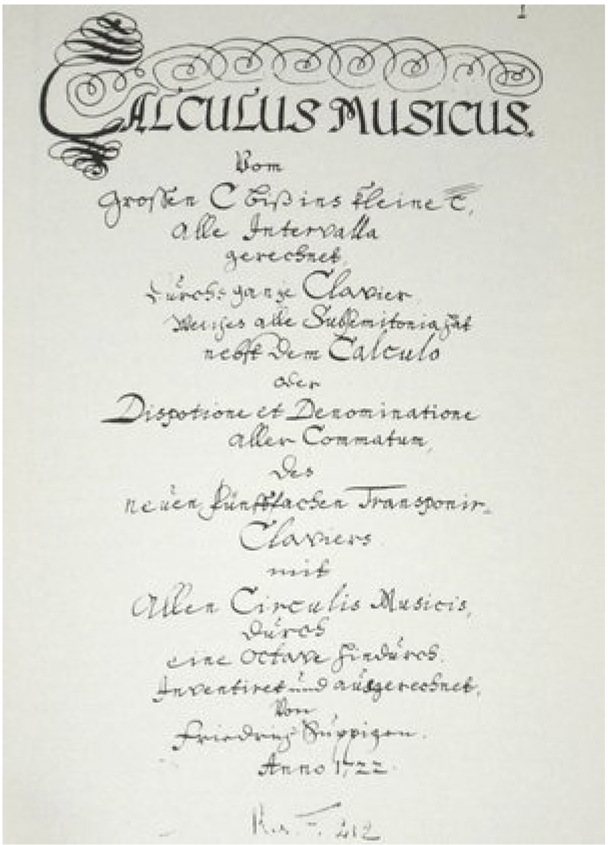
Relationship to Bach

Various features of the cover sheets (Figure 3) suggest that Bach had either seen a copy of Suppig's work or both were influenced by a common source:

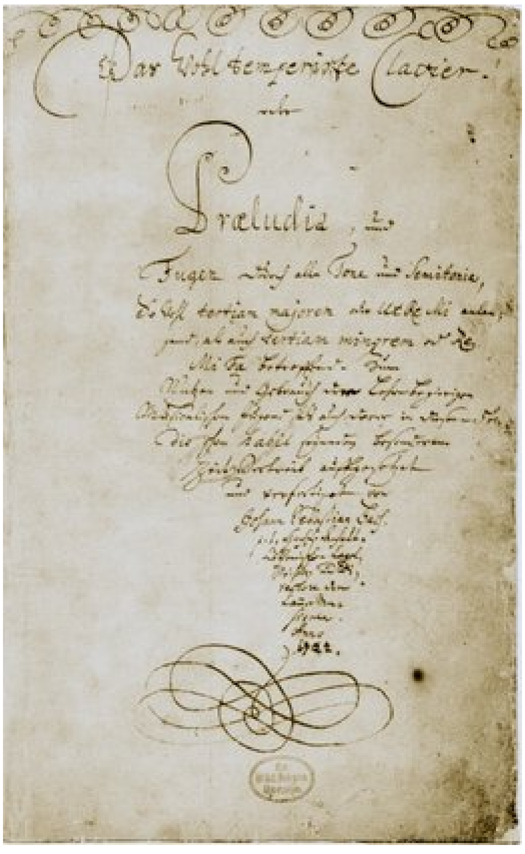
- Tuning is for the one-line octave C to B in terms of beats-per-second [1]
- The spirals contain large loops with one, two and three windings, as well as small terminal loops
- The cover sheets conclude with an identical year, 1722
- The respective spirals, placed at the top of each page above the titles, are elevated in the middle and lowered at the ends with apparent allusion to a circle
- The opening sentences describing the musical compositions are similar:
  - "Musical Labyrinth consisting of a Fantasia through all keys, namely 12 major and 12 minor, together 24 keys."
  - "Preludes and Fugues through all the tones and semitones both as regards the tertia major or Ut Re Mi and as concerns the tertia minor or Re Mi Fa"



LABYRINTHUS
MUSICUS.
Bestehend in
einer
FANTASIA.
durch
alle Tonos,
nemlich
durch
12 duros und 12 molles,
zusammen 24 Tonos.
Dan so wohl auff dem
Clavicymbel
ohne Pedal
als auch
auff der Orgel mit dem Pedale
gespielet werden.
Componiret
von
Friedrich Suppig
Anno 1722.



CALCULUS MUSICUS.
Vom
grossen C bis ins kleine c,
alle Intervalla
gerechnet,
durchs ganze Clavier
welches alle Subsemitonia hat
nebst dem Calculo
oder
Dispositione et Denominatione
aller Commatum,
des
neuen fünfffachen Transponir-
Claviers
mit
allen Circulis Musicis,
durch
eine Octava hindurch.
Inventiret und außgerechnet,
von
Friedrich Suppigen.
Anno 1722.



Das Wohltemperirte Clavier.
oder
Præludia, und
Fugen durch alle Tone und Semitonia,
So wohl tertiam majorem oder Ut Re Mi anlangend,
als auch tertiam minorem oder Re Mi Fa betreffend.
Zum Nutzen und Gebrauch der Lehrbegierigen
Musicalischen Jugend, als auch derer in diesem studio schon
habil seyenden besonderem
ZeitVertreib auffgesetzet
und verfertiget von
Johann Sebastian Bach.
p.t: Hochfürstl.
Anhalt-Cöthenischen
Capel-Meistern
und Directore derer
CammerMusiquen.
Anno
1722.

Figure 3: the three cover sheets

The diagram in *Das Wohltemperirte Clavier* improves on Suppig's diagrammatic schemes in several respects:

- Explicit representation of Cammerton and Cornet-Ton transposes [1]
- Clear separation between outer and inner windings where the large loop implies a tuning operation must be performed and the smaller knot denotes a beat rate
- Common representation for inner windings and end points

The temperament of *Das Wohltemperirte Clavier* [1] and *Calculus musicus* are similarly unequal, with major thirds in the range 394 to 406 cents, minor thirds in the range 294/295 to 305 cents and fifths ranging from 698 to 702 cents. Moreover, both proceed in a progressive manner from best to worst thirds. There are significant differences, however.

As shown in Figure 4 and Figure 5, the *Calculus musicus* temperament has a Werckmeister-like shape with best major thirds on F and C, and worst major third on F# and C#. The best minor third is on A, while the worst are rooted on Eb, Bb and F. On the sharp side of the circle of fifths the progression is gradual, while for the flats the change is more abrupt. The Cornet-Ton temperament R12-2 from *Das Wohltemperirte Clavier* [1] has the opposite shape: rapid change in width of major thirds on the side of the sharps, peaking at E, with gradual change thereafter. The Cammerton transpose R2-1 [1] is shifted by two positions towards the sharps on the circle of fifths with best major thirds on G and D, and worst on F#.

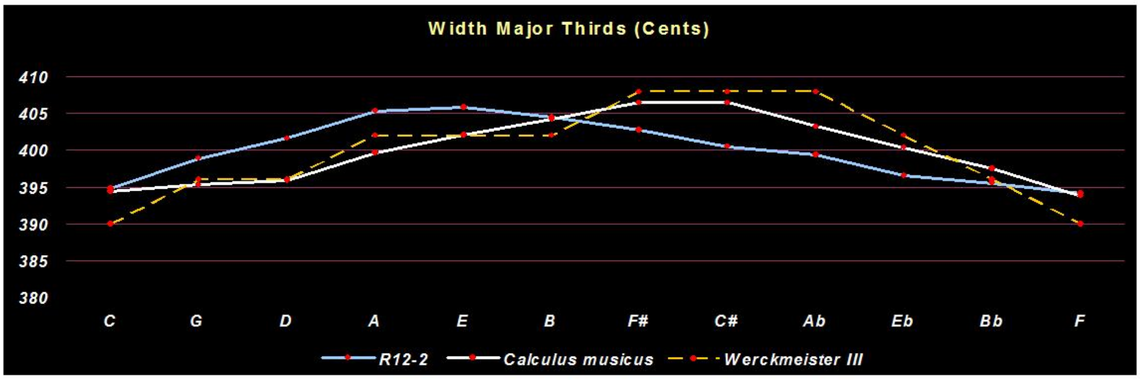


Figure 4: comparison of width of major thirds with Bach Cornet-Ton R12-2 [1]

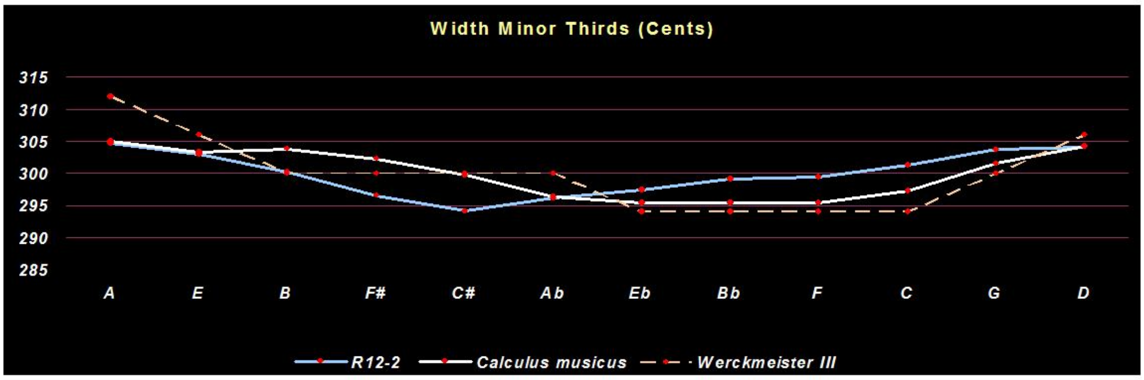


Figure 5: comparison of width of minor thirds with Bach Cornet-Ton R12-2 [1]

Temperament Design

The tempering operations from the *Labyrinthus musicus*, *Calculus musicus* and *Das Wohltemperirte Clavier* can be normalised so that all start at Ab and proceed toward the sharps. Of interest, the total of all tempering operation is 15 in all cases, reflecting the tuning needs of Cornet-Ton pitch. The tempering operations can be written in a dimensionless manner as shown in Table 9. All of the above temperaments are designed to provide an optimally smooth progression from worst to best thirds (both major and minor) across the circle-of-fifths. The thinking that led to the creation of the temperaments can be readily reconstructed from the well known consideration that four consecutive intervals on the circle-of-fifths determine the width of the corresponding major third, while nine determine the minor third. This is shown in Table 10, Table 11, Table 12, Table 13, Table 14 and Table 15.

Temperament	***Tempering of Fifths (from Ab towards sharps)***	***Total***
Labyrinthus musicus	0, 0, 1, 2, 2, 2, 2, 3, 2, 1, 0, 0	15
Labyrinthus musicus (Mirror)[1]	0, 0, 1, 2, 3, 2, 2, 2, 2, 1, 0, 0	15
Calculus musicus	1, 0, 0, 2, 2, 2, 2, 2, 2, 2, 0, 0	15
Calculus musicus (Mirror)[1]	0, 0, 2, 2, 2, 2, 2, 2, 2, 0, 0, 1	15
Das Wohltemperirte Clavier (Cornet-Ton R12-2) [1]	1, 1, 2, 2, 2, 2, 2, 2, 0, 0, 0, 1	15
Das Wohltemperirte Clavier (Cornet-Ton 7-2) [1] (= mirror Cornet-Ton R12-2)	1, 0, 0, 0, 2, 2, 2, 2, 2, 2, 1, 1	15

Table 9: dimensionless representation of tempering operations

Tempering of Fifths	**Tempering Major 3rds**	**Major 3rd**
(0, 0, 1, 2)	3	AbC
0, (0, 1, 2, 2)	5	EbG
0, 0, (1, 2, 2, 2)	7	BbD
0, 0, 1, (2, 2, 2, 2)	8	FA
0, 0, 1, 2, (2, 2, 2, 3)	9	CE
0, 0, 1, 2, 2, (2, 2, 3, 2)	9	GB
0, 0, 1, 2, 2, 2, (2, 3, 2, 1)	8	DF#
0, 0, 1, 2, 2, 2, 2, (3, 2, 1, 0)	6	AC#
0, 0, 1, 2, 2, 2, 2, 3, (2, 1, 0, 0)	3	EAb
0, 0, 1, 2, 2, 2, 2, 3, 2, (1, 0, 0, 0)	1	BEb
0, 0, 1, 2, 2, 2, 2, 3, 2, 1, (0, 0, 0, 0)	0	F#Bb
0, 0, 1, 2, 2, 2, 2, 3, 2, 1, 0, (0, 0, 0, 1)	1	C#F

Table 10: Labyrinthus musicus (monotonic major thirds)

[1] See Appendix.

Tempering of Fifths	Tempering Minor 3rds	Minor 3rd
(0, 0, 1, 2, 2, 2, 2, 3, 2)	14	AbB
0, (0, 1, 2, 2, 2, 2, 3, 2, 1)	15	EbF#
0, 0, (1, 2, 2, 2, 2, 3, 2, 1, 0)	15	BbC#
0, 0, 1, (2, 2, 2, 2, 3, 2, 1, 0, 0)	14	FAb
0, 0, 1, 2, (2, 2, 2, 3, 2, 1, 0, 0, 0)	12	CEb
0, 0, 1, 2, 2, (2, 2, 3, 2, 1, 0, 0, 0, 0)	10	GBb
0, 0, 1, 2, 2, 2, (2, 3, 2, 1, 0, 0, 0, 0, 1)	9	DF
0, 0, 1, 2, 2, 2, 2, (3, 2, 1, 0, 0, 0, 0, 1, 2)	9	AC
0, 0, 1, 2, 2, 2, 2, 3, (2, 1, 0, 0, 0, 0, 1, 2, 2)	8	EG
0, 0, 1, 2, 2, 2, 2, 3, 2, (1, 0, 0, 0, 0, 1, 2, 2, 2)	8	BD
0, 0, 1, 2, 2, 2, 2, 3, 2, 1, (0, 0, 0, 0, 1, 2, 2, 2, 2)	9	F#A
0, 0, 1, 2, 2, 2, 2, 3, 2, 1, 0, (0, 0, 0, 1, 2, 2, 2, 2, 3)	12	C#E

Table 11: Labyrinthus musicus (monotonic minor thirds)

Tempering Fifths	Tempering Major 3rds	Major 3rd
(1, 0, 0, 2)	3	AbC
1, (0, 0, 2, 2)	4	EbG
1, 0, (0, 2, 2, 2)	6	BbD
1, 0, 0, (2, 2, 2, 2)	8	FA
1, 0, 0, 2, (2, 2, 2, 2)	8	CE
1, 0, 0, 2, 2, (2, 2, 2, 2)	8	GB
1, 0, 0, 2, 2, 2, (2, 2, 2, 2)	8	DF#
1, 0, 0, 2, 2, 2, 2, (2, 2, 2, 0)	6	AC#
1, 0, 0, 2, 2, 2, 2, 2, (2, 2, 0, 0)	4	EAb
1, 0, 0, 2, 2, 2, 2, 2, 2, (2, 0, 0, 1)	3	BEb
1, 0, 0, 2, 2, 2, 2, 2, 2, 2, (0, 0, 1, 0)	1	F#Bb
1, 0, 0, 2, 2, 2, 2, 2, 2, 2, 0, (0, 1, 0, 0)	1	C#F

Table 12: *Calculus musicus (monotonic major thirds)*

Tempering Fifths	Tempering Minor 3rds	Minor 3rd
(1, 0, 0, 2, 2, 2, 2, 2, 2)	13	AbB
1, (0, 0, 2, 2, 2, 2, 2, 2, 2)	14	EbF#
1, 0, (0, 2, 2, 2, 2, 2, 2, 2, 0)	14	BbC#
1, 0, 0, (2, 2, 2, 2, 2, 2, 2, 0, 0)	14	FAb
1, 0, 0, 2, (2, 2, 2, 2, 2, 2, 0, 0, 1)	13	CEb
1, 0, 0, 2, 2, (2, 2, 2, 2, 2, 0, 0, 1, 0)	11	GBb
1, 0, 0, 2, 2, 2, (2, 2, 2, 2, 0, 0, 1, 0, 0)	9	DF
1, 0, 0, 2, 2, 2, 2, (2, 2, 2, 0, 0, 1, 0, 0, 2)	9	AC
1, 0, 0, 2, 2, 2, 2, 2, (2, 2, 0, 0, 1, 0, 0, 2, 2)	9	EG
1, 0, 0, 2, 2, 2, 2, 2, 2, (2, 0, 0, 1, 0, 0, 2, 2, 2)	9	BD
1, 0, 0, 2, 2, 2, 2, 2, 2, 2, (0, 0, 1, 0, 0, 2, 2, 2, 2)	9	F#A
1, 0, 0, 2, 2, 2, 2, 2, 2, 2, 0, (0, 1, 0, 0, 2, 2, 2, 2, 2)	11	C#E

Table 13: Calculus musicus (monotonic minor thirds)

Tempering Fifths	Tempering Major 3rds	Major 3rd
(1, 1, 2, 2)	6	AbC
1, (1, 2, 2, 2)	7	EbG
1, 1, (2, 2, 2, 2)	8	BbD
1, 1, 2, (2, 2, 2, 2)	8	FA
1, 1, 2, 2, (2, 2, 2, 2)	8	CE
1, 1, 2, 2, 2, (2, 2, 2, 0)	6	GB
1, 1, 2, 2, 2, 2, (2, 2, 0, 0)	4	DF#
1, 1, 2, 2, 2, 2, 2, (2, 0, 0, 0)	2	AC#
1, 1, 2, 2, 2, 2, 2, 2, (0, 0, 0, 1)	1	EAb
1, 1, 2, 2, 2, 2, 2, 2, 0, (0, 0, 1, 1)	2	BEb
1, 1, 2, 2, 2, 2, 2, 2, 0, 0, (0, 1, 1, 1)	3	F#Bb
1, 1, 2, 2, 2, 2, 2, 2, 0, 0, 0, (1, 1, 1, 2)	5	C#F

Table 14: Das Wohltemperirte Clavier *(monotonic major thirds)*

Tempering Fifths	Tempering Minor 3rds	Minor 3rd
(1, 1, 2, 2, 2, 2, 2, 0)	14	AbB
1, (1, 2, 2, 2, 2, 2, 0, 0)	13	EbF#
1, 1, (2, 2, 2, 2, 2, 0, 0, 0)	12	BbC#
1, 1, 2, (2, 2, 2, 2, 0, 0, 0, 1)	11	FAb
1, 1, 2, 2, (2, 2, 2, 0, 0, 0, 1, 1)	10	CEb
1, 1, 2, 2, 2, (2, 2, 0, 0, 0, 1, 1, 1)	9	GBb
1, 1, 2, 2, 2, 2, (2, 0, 0, 0, 1, 1, 1, 2)	9	DF
1, 1, 2, 2, 2, 2, 2, (0, 0, 0, 1, 1, 1, 2, 2)	9	AC
1, 1, 2, 2, 2, 2, 2, 2, (0, 0, 0, 1, 1, 1, 2, 2, 2)	9	EG
1, 1, 2, 2, 2, 2, 2, 2, 0, (0, 0, 1, 1, 1, 2, 2, 2, 2)	11	BD
1, 1, 2, 2, 2, 2, 2, 2, 0, 0, (0, 1, 1, 1, 2, 2, 2, 2, 2)	13	F#A
1, 1, 2, 2, 2, 2, 2, 2, 0, 0, 0, (1, 1, 1, 2, 2, 2, 2, 2, 2)	15	C#E

Table 15: Das Wohltemperirte Clavier *(monotonic minor thirds)*

Theoretical Approximations

For the purpose of theoretical analysis using cents or other logarithmic measures, each narrowing by one beat per second at Cornet-Ton pitch can be approximated by 1/15 root of the Pythagorean Comma (Table 16). A courser approximation can be made using 1/12 Pythagorean Comma, introducing a wide wolf fifth of 3/12 Pythagorean Comma to compensate for the over-tempering (Table 17). Note that the latter approximation compromises the monotonic behaviour of major and minor thirds.

Approximated Temperament	***Theoretical Approximation***
Labyrinthus musicus	Ab 0 Eb 0 Bb -1/15 F -2/15 C -2/15 G -2/15 D -2/15 A -3/15 E -2/15 B -1/15 F# 0 C# 0 Ab
Labyrinthus musicus (Mirror)[2]	Ab 0 Eb 0 Bb -1/15 F -2/15 C -3/15 G -2/15 D -2/15 A -2/15 E -2/15 B -1/15 F# 0 C# 0 Ab
Calculus musicus	Ab -1/15 Eb 0 Bb 0 F -2/15 C -2/15 G -2/15 D -2/15 A -2/15 E -2/15 B -2/15 F# 0 C# 0 Ab
Calculus musicus (Mirror)[2]	Ab 0 Eb 0 Bb -2/15 F -2/15 C -2/15 G -2/15 D -2/15 A -2/15 E -2/15 B 0 F# 0 C# -1/15 Ab

Table 16: approximation of beat per second by 1/15 Pythagorean Comma

Approximated Temperament	***Theoretical Approximation***
Labyrinthus musicus	Ab 0 Eb 0 Bb -1/12 F -2/12 C -2/12 G -2/12 D -2/12 A -3/12 E -2/12 B -1/12 F# 0 C# 3/12 Ab
Labyrinthus musicus (Mirror)[2]	Ab 3/12 Eb 0 Bb -1/12 F -2/12 C -3/12 G -2/12 D -2/12 A -2/12 E -2/12 B -1/12 F# 0 C# 0 Ab
Calculus musicus	Ab -1/12 Eb 0 Bb 0 F -2/12 C -2/12 G -2/12 D -2/12 A -2/12 E -2/12 B -2/12 F# 0 C# 3/12 Ab
Calculus musicus (Mirror)[2]	Ab 3/12 Eb 0 Bb -2/12 F -2/12 C -2/12 G -2/12 D -2/12 A -2/12 E -2/12 B 0 F# 0 C# -1/12 Ab

Table 17: approximation of beat per second by 1/12 Pythagorean Comma

References

[1] John Charles Francis, “The Esoteric Keyboard Temperaments of J.S. Bach”, Eunomios, February 2005. PDF-file download: http://www.eunomios.org/

[2] Friedrich Suppig: Labyrinthus musicus & Calculus musicus. Facsimile of the manuscripts Paris, Bibliothèque du Conservatoire Rés. F 211–212 (Dated Dresden, 24 June 1722), Diapason Press, Ed. Rudolf Rasch, 1990.

[3] Andreas Werckmeister: Musicalische Temperatur (Quedlinburg 1691), Diapason Press, Ed. Rudolf Rasch, 1983.

[3] Bach Tuning: http://bach.tuning.googlepages.com/

Acknowledgements

I am grateful to Andreas Sparschuh for pointing me to Friedrich Suppig’s work and for helpful remarks regarding its interpretation. I am also indebted to Thomas Braatz for assistance with relevant aspects of Bach’s handwriting. Finally, I would like to acknowledge the support of my wife, Gabi.

[2] See Appendix.

Appendix

Mirror Temperaments

If it is assumed for arguments sake that the Ab in the Labyrinthus musicus depicts G#, then the following reading results:

Eb-0-Bb-1-F-2-C-3-G-2-D-2-A-2-E-2-B-1-F#[-0-C#-0]-G#-x

Harmonic Relations
3f0 - 2f7 = 3
3f7 - 2f14 = 2
2f2 - f14 = 0
3f2 - 2f9 = 2
3f9 - 2f16 = 2
2f4 - f16 = 0
3f4 - 2f11 = 2
3f11 - 2f18 = 1
2f6 - f18 = 0
3f6 - 2f13 = 0
2f1 - f13 = 0
3f1 - 2f8 = 0
3f8 - 2f15 = x
2f3 - f15 = 0
3f3 - 2f10 = 0
3f10 - 2f17 = 1
2f5 - f17 = 0
3f5 - 2f12 = 2
2f0 - f12 = 0

The results show that tuning takes place in the one-line octave from C to B, an identical tuning range to the Bach spiral [1]. In the case where the beat rate of G#Eb is pure, this is Cornet-Ton pitch.

Pitch			
Pitch of A	*415 Hz*	*474.577 Hz*	*465 Hz*
Beat Rate G#Eb	-2.28347 Hz (wide)	0 Hz (pure)	-0.367054 Hz (wide)

The analysis continues for the case where G#Eb is pure.

Cents	
C	0
C#	94.4217
D	195.059
Eb	298.332
E	392.892
F	501.098
F#	592.467
G	695.833
Ab	796.377
A	893.37
Bb	1000.29
B	1091.6

Width of Thirds		
Root	*Minor 3rd*	*Major 3rd*
C	298	393
G	304	396
D	306	397
A	307	401
E	303	403
B	303	407
F#	301	408
C#	298	407
Ab	295	404
Eb	294	398
Bb	294	395
F	295	392

The 5^{ths} are in the range 696-702 cents. The interval C-G is 696 cents.

For the *Calculus musicus* the mirror reading is:

Ab-1-C#-0-F#-0-B-2-E-2-A-2-D-2-G-2-C-2-F-2-Bb[-0-Eb-0-]

Harmonic Relations
3f0 - 2f7 = 2
3f7 - 2f14 = 2
2f2 - f14 = 0
3f2 - 2f9 = 2
3f9 - 2f16 = 2
2f4 - f16 = 0
3f4 - 2f11 = 2
3f11 - 2f18 = 0
2f6 - f18 = 0
3f6 - 2f13 = 0
2f1 - f13 = 0
3f1 - 2f8 = 1
3f8 - 2f15 = 0
2f3 - f15 = 0
3f3 - 2f10 = 0
3f10 - 2f17 = 2
2f5 - f17 = 0
3f5 - 2f12 = 2
2f0 - f12 = 0

The results show that tuning takes place in the Cornet-Ton one-line octave from C to B at A=473.661 Hz. The cents values for each semitone are:

Cents	
C	0
C#	97.5137
D	197.085
Eb	299.491
E	394.905
F	501.108
F#	595.559
G	697.863
Ab	797.536
A	895.389
Bb	1001.45
B	1093.6

Width of Thirds		
Root	*Minor 3rd*	*Major 3rd*
C	299	395
G	304	396
D	304	398
A	305	402
E	303	403
B	303	406
F#	300	406
C#	297	404
Ab	296	402
Eb	296	398
Bb	296	396
F	296	394

Comparisons